

ON AN OPERATOR PRESERVING INEQUALITIES BETWEEN POLYNOMIALS

N. A. RATHER, SUHAIL GULZAR

Department of Mathematics, University of Kashmir, Srinagar 190006, India

dr.narather@gmail.com, sgmattoo@gmail.com

Abstract. Let \mathcal{P}_n denote the space of all complex polynomials $P(z) = \sum_{j=0}^n a_j z^j$ of degree n and \mathcal{B}_n a family of operators that maps \mathcal{P}_n into itself. In this paper, we consider a problem of investigating the dependence of

$$\left| B[P \circ \sigma](z) - \alpha B[P \circ \rho](z) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} B[P \circ \rho](z) \right|$$

on the maximum and minimum modulus of $|P(z)|$ on $|z| = k$ for arbitrary real or complex numbers $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq k, \sigma(z) = Rz, \rho(z) = rz$ and establish certain sharp operator preserving inequalities between polynomials, from which a variety of interesting results follow as special cases.

Keywords: Polynomials; Inequalities in the complex domain; \mathcal{B}_n -operator.

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1. Introduction

Let \mathcal{P}_n denote the space of all complex polynomials $P(z) = \sum_{j=0}^n a_j z^j$ of degree n . A famous result known as Bernstein's inequality (for reference, see [8, p.531], [10, p.508] or [11] states that if $P \in \mathcal{P}_n$, then

$$(1) \quad \max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|,$$

whereas concerning the maximum modulus of $P(z)$ on the circle $|z| = R > 1$, we have

$$(2) \quad \max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)|, \quad R \geq 1.$$

(for reference, see [7, p.442] or [8, vol.I, p.137]).

If we restrict ourselves to the class of polynomials $P \in \mathcal{P}_n$ having no zero in $|z| < 1$, then inequalities (1) and (2) can be respectively replaced by

$$(3) \quad \max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|,$$

and

$$(4) \quad \max_{|z|=R} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|, \quad R \geq 1.$$

Inequality (3) was conjectured by Erdős and later verified by Lax [5], whereas inequality (4) is due to Ankey and Ravilin [1]. Aziz and Dawood [2] further improved inequalities (3) and (4) under the same hypothesis and proved that,

$$(5) \quad \max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\},$$

$$(6) \quad \max_{|z|=R} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)| - \frac{R^n - 1}{2} \min_{|z|=1} |P(z)|, \quad R \geq 1.$$

As a compact generalization of Inequalities (1) and (2), Aziz and Rather [3] have shown that if $P \in \mathcal{P}_n$ then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > 1$ and $|z| \geq 1$,

$$(7) \quad \begin{aligned} & \left| P(Rz) - \alpha P(z) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} P(z) \right| \\ & \leq |z|^n \left| R^n - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| \max_{|z|=1} |P(z)|. \end{aligned}$$

The result is sharp and equality in (7) holds for the polynomial $P(z) = az^n$, $a \neq 0$.

As a corresponding compact generalization of Inequalities (3) and (4), they [3] have also shown that if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > 1$ and $|z| \geq 1$,

$$(8) \quad \begin{aligned} & \left| P(Rz) - \alpha P(z) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} P(z) \right| \\ & \leq \frac{1}{2} \left[\left| R^n - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| |z|^n \right. \\ & \quad \left. + \left| 1 - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| \right] \max_{|z|=1} |P(z)|. \end{aligned}$$

The result is best possible and equality in (8) holds for $P(z) = az^n + b$, $|a| = |b|$.

Q. I. Rahman [9] (see also Rahman and Schmeisser [10, p. 538]) introduced a class \mathcal{B}_n of operators B that carries a polynomial $P \in \mathcal{P}_n$ into

$$(9) \quad B[P](z) = \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) \frac{P'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!},$$

where λ_0, λ_1 and λ_2 are such that all the zeros of

$$(10) \quad U(z) = \lambda_0 + n\lambda_1 z + \frac{n(n-1)}{2} \lambda_2 z^2$$

lie in half plane $|z| \leq |z - n/2|$.

As a generalization of the inequalities (1) and (3), Q. I. Rahman [9, inequalities 5.2 and 5.3] proved that if $P \in \mathcal{P}_n$, then

$$(11) \quad |B[P](z)| \leq |B[z^n]| \underset{|z|=1}{Max} |P(z)|, \quad \text{for } |z| \geq 1,$$

and if $P \in \mathcal{P}_n$, $P(z) \neq 0$ in $|z| < 1$, then

$$(12) \quad |B[P](z)| \leq \frac{1}{2} \{ |B[z^n]| + |\lambda_0| \} \underset{|z|=1}{Max} |P(z)|, \quad \text{for } |z| \geq 1,$$

where $B \in \mathcal{B}_n$.

In this paper, we denote for any complex functions $P, \rho : \mathbb{C} \rightarrow \mathbb{C}$ the composite function of P and ρ , defined by $(P \circ \rho)(z) = P(\rho(z))$ ($z \in \mathbb{C}$), as $P \circ \rho$.

1. Preliminaries

For the proof of our results, we need the following Lemmas.

Lemma 1.1. *If $P \in \mathcal{P}_n$ and $P(z)$ have all its zeros in $|z| \leq k$ where $k \geq 0$, then for every $R \geq r$, $Rr \geq k^2$ and $|z| = 1$, we have*

$$|P(Rz)| \geq \left(\frac{R+k}{r+k} \right)^n |P(rz)|.$$

The above is due to Aziz and Zargar [4]. The next lemma follows from Corollary 18.3 of [6, p. 86].

Lemma 1.2. *If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k$, where $k > 0$ then all the zeros of $B[P](z)$ also lie in $|z| \leq k$.*

Lemma 1.3. *If $P \in \mathcal{P}_n$ and $P(z)$ have no zero in $|z| < k$, where $k > 0$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$,*

$$(13) \quad \begin{aligned} & \left| B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[P \circ \rho](z) \right| \\ & \leq k^n |B[Q \circ \tau](z)| + \Phi_k(R, r, \alpha, \beta) |B[Q \circ \eta](z)| \end{aligned}$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$, $\sigma(z) = Rz$, $\rho(z) = rz$, $\tau(z) = Rz/k^2$, $\eta(z) = rz/k^2$ and

$$(14) \quad \Phi_k(R, r, \alpha, \beta) = \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} - \alpha.$$

Proof. By hypothesis, the polynomial $P(z)$ does not vanish in $|z| < k$. Therefore, all the zeros of polynomial $Q(z/k^2)$ lie in $|z| < k$. As

$$|k^n Q(z/k^2)| = |P(z)| \quad \text{for } |z| = k,$$

applying Theorem 2.1 to $P(z)$ with $F(z)$ replaced by $k^n Q(z/k^2)$, we get for arbitrary real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$,

$$\begin{aligned} & \left| B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[P \circ \rho](z) \right| \\ & \leq k^n |B[Q \circ \tau](z)| + \Phi_k(R, r, \alpha, \beta) |B[Q \circ \eta](z)|, \end{aligned}$$

This proves Lemma 1.3. □

Lemma 1.4. *If $P \in \mathcal{P}_n$ and $Q(z) = z^n \overline{P(1/\bar{z})}$ then for $\alpha, \beta \in \mathbb{C}$, with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq k, k \leq 1$ and $|z| \geq 1$,*

$$(15) \quad \begin{aligned} & \left| B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[P \circ \rho](z) \right| \\ & + k^n \left| B[Q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta) B[Q \circ \eta](z) \right| \\ & \leq \left\{ |\lambda_0| |1 + \Phi_k(R, r, \alpha, \beta)| + \frac{|B[z^n]|}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| \right\} \max_{|z|=k} |P(z)| \end{aligned}$$

where $\sigma(z) = Rz$, $\rho(z) = rz$, $\tau(z) = Rz/k^2$, $\eta(z) = rz/k^2$ and $\Phi_k(R, r, \alpha, \beta)$ is given by (14).

Proof. Let $M = \max_{|z|=k} |P(z)|$, then by Rouché's theorem, the polynomial $F(z) = P(z) - \mu M$ does not vanish in $|z| < k$ for every $\mu \in \mathbb{C}$ with $|\mu| > 1$. Applying Lemma 1.3 to polynomial $F(z)$, we get for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| \geq 1$,

$$|B[F \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta)B[F \circ \rho](z)| \leq k^n |B[H \circ \tau](z) + \Phi_k(R, r, \alpha, \beta)B[H \circ \eta](z)|,$$

where $H(z) = z^n \overline{F(1/\bar{z})} = Q(z) - \bar{\mu} M z^n$. Replacing $F(z)$ by $P(z) - \mu M$ and $H(z)$ by $Q(z) - \bar{\mu} M z^n$, we have for $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| \geq 1$,

$$\begin{aligned} & |B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta)B[P \circ \rho](z) - \mu \lambda_0 (1 + \Phi_k(R, r, \alpha, \beta)) M| \\ & \leq k^n \left| B[Q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta)B[Q \circ \eta](z) \right. \\ & \quad \left. - \frac{\bar{\mu}}{k^{2n}} (R^n + r^n \Phi_k(R, r, \alpha, \beta)) M B[z^n] \right| \end{aligned} \quad (16)$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$.

Now choosing argument of μ in the right hand side of inequality (16) such that

$$\begin{aligned} & k^n \left| B[Q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta)B[Q \circ \eta](z) - \frac{\bar{\mu}}{k^{2n}} (R^n + r^n \Phi_k(R, r, \alpha, \beta)) M B[z^n] \right| \\ & = \frac{|\bar{\mu}|}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| |B[z^n]| M - k^n |B[Q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta)B[Q \circ \eta](z)| \end{aligned}$$

which is possible by applying Corollary 2.3 to polynomial $Q(z/k^2)$ and using the fact $\max_{|z|=k} |Q(z/k^2)| = M/k^n$, we get for $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| \geq 1$,

$$\begin{aligned} & |B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta)B[P \circ \rho](z)| - |\mu \lambda_0| (1 + \Phi_k(R, r, \alpha, \beta)) M \\ & \leq \frac{|\bar{\mu}|}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| |B[z^n]| M - k^n |B[Q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta)B[Q \circ \eta](z)| \end{aligned}$$

Equivalently for $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| \geq 1$,

$$\begin{aligned} & |B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta)B[P \circ \rho](z)| + k^n |B[Q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta)B[Q \circ \eta](z)| \\ & \leq |\mu| \left\{ |\lambda_0| (1 + \Phi_k(R, r, \alpha, \beta)) + \frac{1}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| |B[z^n]| \right\} M \end{aligned}$$

Letting $|\mu| \rightarrow 1$, we get the conclusion of Lemma 1.4 and this completes proof of Lemma 1.4. \square

2. Main results

Theorem 2.1. *If $F \in \mathcal{P}_n$ and $F(z)$ has all its zeros in the disk $|z| \leq k$ where $k > 0$ and $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)| \leq |F(z)| \quad \text{for } |z| = k,$$

then for $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq k$ and $|z| \geq 1$,

$$(17) \quad \begin{aligned} & |B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta)B[P \circ \rho](z)| \\ & \leq |B[F \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta)B[F \circ \rho](z)| \end{aligned}$$

where $\sigma(z) = Rz, \rho(z) = rz$ and

$$(18) \quad \Phi_k(R, r, \alpha, \beta) = \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} - \alpha.$$

The result is best possible and the equality holds for the polynomial $P(z) = e^{i\gamma}F(z)$ where $\gamma \in \mathbb{R}$.

Proof of Theorem 2.1. Since polynomial $F(z)$ of degree n has all its zeros in $|z| \leq k$ and $P(z)$ is a polynomial of degree at most n such that

$$(19) \quad |P(z)| \leq |F(z)| \quad \text{for } |z| = k,$$

therefore, if $F(z)$ has a zero of multiplicity s at $z = ke^{i\theta_0}$, $0 \leq \theta_0 < 2\pi$, then $P(z)$ has a zero of multiplicity at least s at $z = ke^{i\theta_0}$. If $P(z)/F(z)$ is a constant, then inequality (17) is obvious. We now assume that $P(z)/F(z)$ is not a constant, so that by the maximum modulus principle, it follows that

$$|P(z)| < |F(z)| \quad \text{for } |z| > k.$$

Suppose $F(z)$ has m zeros on $|z| = k$ where $0 \leq m < n$, so that we can write

$$F(z) = F_1(z)F_2(z)$$

where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z| = k$ and $F_2(z)$ is a polynomial of degree exactly $n - m$ having all its zeros in $|z| < k$. This implies with the help of inequality (19) that

$$P(z) = P_1(z)F_1(z)$$

where $P_1(z)$ is a polynomial of degree at most $n - m$. Again, from inequality (19), we have

$$|P_1(z)| \leq |F_2(z)| \text{ for } |z| = k$$

where $F_2(z) \neq 0$ for $|z| = k$. Therefore for every real or complex number λ with $|\lambda| > 1$, a direct application of Rouché's theorem shows that the zeros of the polynomial $P_1(z) - \lambda F_2(z)$ of degree $n - m \geq 1$ lie in $|z| < k$ hence the polynomial

$$G(z) = F_1(z)(P_1(z) - \lambda F_2(z)) = P(z) - \lambda F(z)$$

has all its zeros in $|z| \leq k$ with at least one zero in $|z| < k$, so that we can write

$$G(z) = (z - te^{i\delta})H(z)$$

where $t < k$ and $H(z)$ is a polynomial of degree $n - 1$ having all its zeros in $|z| \leq k$. Applying Lemma 1.1 to the polynomial $H(z)$, we obtain for every $R > r \geq k$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned} |G(Re^{i\theta})| &= |Re^{i\theta} - te^{i\delta}| |H(Re^{i\theta})| \\ &\geq |Re^{i\theta} - te^{i\delta}| \left(\frac{R+k}{k+r} \right)^{n-1} |H(re^{i\theta})|, \\ &= \left(\frac{R+k}{k+r} \right)^{n-1} \frac{|Re^{i\theta} - te^{i\delta}|}{|re^{i\theta} - te^{i\delta}|} |(re^{i\theta} - te^{i\delta})H(re^{i\theta})|, \\ &\geq \left(\frac{R+k}{k+r} \right)^{n-1} \left(\frac{R+t}{r+t} \right) |G(re^{i\theta})|. \end{aligned}$$

This implies for $R > r \geq k$ and $0 \leq \theta < 2\pi$,

$$(20) \quad \left(\frac{r+t}{R+t} \right) |G(Re^{i\theta})| \geq \left(\frac{R+k}{k+r} \right)^{n-1} |G(re^{i\theta})|.$$

Since $R > r \geq k$ so that $G(Re^{i\theta}) \neq 0$ for $0 \leq \theta < 2\pi$ and $\frac{r+k}{k+R} > \frac{r+t}{R+t}$, from inequality (20), we obtain

$$(21) \quad |G(Re^{i\theta})| > \left(\frac{R+k}{k+r}\right)^n |G(re^{i\theta})|, \quad R > r \geq k \quad \text{and} \quad 0 \leq \theta < 2\pi.$$

Equivalently,

$$|G(Rz)| > \left(\frac{R+k}{k+r}\right)^n |G(rz)|$$

for $|z| = 1$ and $R > r \geq k$. Hence for every real or complex number α with $|\alpha| \leq 1$ and $R > r \geq k$, we have

$$(22) \quad \begin{aligned} |G(Rz) - \alpha G(rz)| &\geq |G(Rz)| - |\alpha| |G(rz)| \\ &> \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} |G(rz)|, \quad \text{for } |z| = 1. \end{aligned}$$

Also, inequality (21) can be written in the form

$$(23) \quad |G(re^{i\theta})| < \left(\frac{k+r}{R+k}\right)^n |G(Re^{i\theta})|$$

for every $R > r \geq k$ and $0 \leq \theta < 2\pi$. Since $G(Re^{i\theta}) \neq 0$ and $\left(\frac{k+r}{R+k}\right)^n < 1$, from inequality (23), we obtain for $0 \leq \theta < 2\pi$ and $R > r \geq k$,

$$|G(re^{i\theta})| < |G(Re^{i\theta})|.$$

That is,

$$|G(rz)| < |G(Rz)| \quad \text{for } |z| = 1.$$

Since all the zeros of $G(Rz)$ lie in $|z| \leq (k/R) < 1$, a direct application of Rouché's theorem shows that the polynomial $G(Rz) - \alpha G(rz)$ has all its zeros in $|z| < 1$ for every real or complex number α with $|\alpha| \leq 1$. Applying Rouché's theorem again, it follows from (22) that for arbitrary real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq k$, all the zeros of the polynomial

$$\begin{aligned} T(z) &= G(Rz) - \alpha G(rz) + \beta \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} G(rz) \\ &= \left[P(Rz) - \alpha P(rz) + \beta \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} P(rz) \right] \\ &\quad - \lambda \left[F(Rz) - \alpha F(rz) + \beta \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} F(rz) \right] \end{aligned}$$

lie in $|z| < 1$.

Applying Lemma 1.3 to the polynomial $T(z)$ and noting that B is a linear operator, it follows that all the zeros of polynomial

$$B[T](z) = \left[B[P \circ \sigma](z) - \alpha B[P \circ \rho](z) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} B[P \circ \rho](z) \right] \\ - \lambda \left[B[F \circ \sigma](z) - \alpha B[F \circ \rho](z) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} [F \circ \rho](z) \right]$$

lie in $|z| < 1$. This implies

$$(24) \quad \begin{aligned} & |B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[P \circ \rho](z)| \\ & \leq |B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[P \circ \rho](z)|, \end{aligned}$$

for $|z| \geq 1$ and $R > r \geq k$. If inequality (24) is not true, then there a point $z = z_0$ with $|z_0| \geq 1$ such that

$$\begin{aligned} & \left| \{B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[P \circ \rho](z)\}_{z=z_0} \right| \\ & \geq \left| \{B[F \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[F \circ \rho](z)\}_{z=z_0} \right|, \end{aligned}$$

But all the zeros of $F(Rz)$ lie in $|z| < (k/R) < 1$, therefore, it follows (as in case of $G(z)$) that all the zeros of $F(Rz) - \alpha F(rz) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} F(rz)$ lie in $|z| < 1$. Hence, by Lemma 1.3,

$$\{B[F \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[F \circ \rho](z)\}_{z=z_0} \neq 0$$

with $|z_0| \geq 1$. We take

$$\lambda = \frac{\{B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[P \circ \rho](z)\}_{z=z_0}}{\{B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[P \circ \rho](z)\}_{z=z_0}},$$

then λ is a well defined real or complex number with $|\lambda| > 1$ and with this choice of λ , we obtain $\{B[T](z)\}_{z=z_0} = 0$ where $|z_0| \geq 1$. This contradicts the fact that all the zeros of $B[T(z)]$ lie in $|z| < 1$. Thus (24) holds for $|\alpha| \leq 1$, $|\beta| \leq 1$, $|z| \geq 1$, and $R > r \geq k$.

□

For $\alpha = 0$ in Theorem 2.1, we obtain the following.

Corollary 2.2. *If $F \in \mathcal{P}_n$ and $F(z)$ has all its zeros in the disk $|z| \leq k$, where $k > 0$ and $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)| \leq |F(z)| \text{ for } |z| = k,$$

then for $|\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$,

$$(25) \quad \begin{aligned} & \left| B[P \circ \sigma](z) + \beta \left(\frac{R+k}{k+r} \right)^n B[P \circ \rho](z) \right| \\ & \leq \left| B[F \circ \sigma](z) + \beta \left(\frac{R+k}{k+r} \right)^n B[F \circ \rho](z) \right| \end{aligned}$$

where $\sigma(z) = Rz$, $\rho(z) = rz$. The result is sharp, and the equality holds for the polynomial $P(z) = e^{i\gamma} F(z)$ where $\gamma \in \mathbb{R}$.

If we choose $F(z) = z^n M/k^n$, where $M = \max_{|z|=k} |P(z)|$ in Theorem 2.1, we get the following result.

Corollary 2.3. *If $P \in \mathcal{P}_n$ then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k > 0$ and $|z| = 1$,*

$$(26) \quad \begin{aligned} & \left| B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[P \circ \rho](z) \right| \\ & \leq \frac{1}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| |B[z^n]| \max_{|z|=k} |P(z)| \end{aligned}$$

where $\sigma(z) = Rz$, $\rho(z) = rz$ and $\Phi_k(R, r, \alpha, \beta)$ is given by (18). The result is best possible and equality in (26) holds for $P(z) = az^n$, $a \neq 0$.

Next, we take $P(z) = z^n m/k^n$, where $m = \min_{|z|=k} |P(z)|$ in Theorem 2.1, we get the following result.

Corollary 2.4. *If $F \in \mathcal{P}_n$ and $F(z)$ have all its zeros in the disk $|z| \leq k$, where $k > 0$ then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k > 0$*

$$(27) \quad \begin{aligned} & \min_{|z|=1} \left| B[F \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[F \circ \rho](z) \right| \\ & \geq \frac{|B[z^n]|}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| \min_{|z|=k} |P(z)|, \end{aligned}$$

where $\sigma(z) = Rz$, $\rho(z) = rz$ and $\Phi_k(R, r, \alpha, \beta)$ is given by (18). The result is Sharp.

If we take $\beta = 0$ in (26), we get the following result.

Corollary 2.5. *If $P \in \mathcal{P}_n$ then for $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $R > r \geq k > 0$ and $|z| \geq 1$,*

$$(28) \quad |B[P \circ \sigma](z) - \alpha B[P \circ \rho](z)| \leq \frac{1}{k^n} |R^n - \alpha r^n| |B[z^n]| \underset{|z|=k}{Max} |P(z)|$$

where $\sigma(z) = Rz$, $\rho(z) = rz$. The result is best possible as shown by $P(z) = az^n$, $a \neq 0$.

For polynomials $P \in \mathcal{P}_n$ having no zero in $|z| < k$, we establish the following result which leads to a compact generalization of inequality (3),(4),(8) and (12).

Theorem 2.6. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in the disk $|z| < k$, where $k \leq 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k > 0$ and $|z| \geq 1$,*

$$(29) \quad \begin{aligned} & |B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[P \circ \rho](z)| \\ & \leq \frac{1}{2} \left[\frac{|B[z^n]|}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| + |1 + \Phi_k(R, r, \alpha, \beta)| |\lambda_0| \right] \underset{|z|=k}{Max} |P(z)| \end{aligned}$$

where $\sigma(z) = Rz$, $\rho(z) = rz$ and $\Phi_k(R, r, \alpha, \beta)$ is given by (18).

Proof of Theorem 2.6. Since $P(z)$ does not vanish in $|z| < k$, $k \leq 1$, by Lemma 1.3, we have for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > 1$ and $|z| \geq 1$,

$$(30) \quad \begin{aligned} & |B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[P \circ \rho](z)| \\ & \leq k^n |B[Q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta) B[Q \circ \eta](z)|, \end{aligned}$$

where $\sigma(z) = Rz$, $\rho(z) = rz$, $\tau(z) = Rz/k^2$, $\eta(z) = rz/k^2$ and $Q(z) = z^n \overline{P(1/\bar{z})}$. Inequality (30) in conjunction with Lemma 1.4 gives for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$,

$$\begin{aligned} & 2|B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[P \circ \rho](z)| \\ & \leq |B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[P \circ \rho](z)| + k^n |B[Q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta) B[Q \circ \eta](z)| \\ & \leq \left\{ |\lambda_0| |1 + \Phi_k(R, r, \alpha, \beta)| + \frac{|B[z^n]|}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| \right\} \underset{|z|=k}{Max} |P(z)|. \end{aligned}$$

This completes the proof of Theorem 2.6.

□

We finally prove the following result, which is the refinement of Theorem 2.6.

Theorem 2.7. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in the disk $|z| < k$, where $k \leq 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k > 0$ and $|z| = 1$,*

$$\begin{aligned}
 & \left| B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[P \circ \rho](z) \right| \\
 & \leq \frac{1}{2} \left[\left\{ \frac{|B[z^n]|}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| + |1 + \Phi_k(R, r, \alpha, \beta)| |\lambda_0| \right\} \max_{|z|=k} |P(z)| \right. \\
 (31) \quad & \left. - \left\{ \frac{|B[z^n]|}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| - |1 + \Phi_k(R, r, \alpha, \beta)| |\lambda_0| \right\} \min_{|z|=k} |P(z)| \right]
 \end{aligned}$$

where $\sigma(z) = Rz$, $\rho(z) = rz$ and $\Phi_k(R, r, \alpha, \beta)$ is given by (18).

Proof of Theorem 2.7. Let $m = \min_{|z|=k} |P(z)|$. If $P(z)$ has a zero on $|z| = k$, then the result follows from Theorem 2.6. We assume that $P(z)$ has all its zeros in $|z| > k$ where $k \leq 1$ so that $m > 0$. Now for every δ with $|\delta| < 1$, it follows by Rouché's theorem $h(z) = P(z) - \delta m$ does not vanish in $|z| < k$. Applying Lemma 1.3 to the polynomial $h(z)$, we get for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$

$$|B[h \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[h \circ \rho](z)| \leq k^n |B[q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta) B[q \circ \eta](z)|,$$

where $\sigma(z) = Rz$, $\rho(z) = rz$, $\tau(z) = Rz/k^2$, $\eta(z) = rz/k^2$ and $q(z) = z^n \overline{h(1/\bar{z})} = z^n \overline{P(1/\bar{z})} - \bar{\delta} m z^n$. Equivalently,

$$\begin{aligned}
 & |B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta) B[P \circ \rho](z) - \delta \lambda_0 (1 + \Phi_k(R, r, \alpha, \beta)) m| \\
 & \leq k^n \left| B[Q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta) B[Q \circ \eta](z) \right. \\
 (32) \quad & \left. - \frac{\bar{\delta}}{k^{2n}} (R^n + r^n \Phi_k(R, r, \alpha, \beta)) m B[z^n] \right|
 \end{aligned}$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$. Since all the zeros of $Q(z/k^2)$ lie in $|z| \leq k$, $k \leq 1$ by Corollary 2.4 applied to $Q(z/k^2)$, we have for $R > 1$ and $|z| = 1$,

$$\begin{aligned}
 & |B[Q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta)B[Q \circ \eta](z)| \\
 & \geq \frac{1}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| |B[z^n]| \min_{|z|=k} Q(z/k^2) \\
 (33) \quad & = \frac{1}{k^{2n}} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| |B[z^n]| m.
 \end{aligned}$$

Now, choosing the argument of δ on the right hand side of inequality (32) such that

$$\begin{aligned}
 & k^n \left| B[Q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta)B[Q \circ \eta](z) - \frac{\bar{\delta}}{k^{2n}} (R^n + r^n \Phi_k(R, r, \alpha, \beta)) m B[z^n] \right| \\
 & = k^n |B[Q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta)B[Q \circ \eta](z)| - \frac{1}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| |B[z^n]| m.
 \end{aligned}$$

for $|z| = 1$, which is possible by inequality (33). We get for $|z| = 1$,

$$\begin{aligned}
 & |B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta)B[P \circ \rho](z)| - |\delta| |\lambda_0| |1 + \Phi_k(R, r, \alpha, \beta)| m \\
 & \leq k^n |B[Q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta)B[Q \circ \eta](z)| \\
 (34) \quad & - \frac{|\delta|}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| |B[z^n]| m.
 \end{aligned}$$

Equivalently for $|z| = 1$, $R > r \geq k$, we have

$$\begin{aligned}
 & |B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta)B[P \circ \rho](z)| - k^n |B[Q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta)B[Q \circ \eta](z)| \\
 (35) \quad & \leq |\delta| \left\{ |\lambda_0| |1 + \Phi_k(R, r, \alpha, \beta)| - \frac{1}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| |B[z^n]| \right\} m.
 \end{aligned}$$

Letting $|\delta| \rightarrow 1$ in inequality (35), we obtain for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq k$ and $|z| = 1$,

$$\begin{aligned}
 & |B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta)B[P \circ \rho](z)| - k^n |B[Q \circ \tau](z) + \Phi_k(R, r, \alpha, \beta)B[Q \circ \eta](z)| \\
 (36) \quad & \leq \left\{ |\lambda_0| |1 + \Phi_k(R, r, \alpha, \beta)| - \frac{1}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| |B[z^n]| \right\} m.
 \end{aligned}$$

Inequality (36) in conjunction with Lemma 1.4 gives for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > 1$ and $|z| = 1$,

$$\begin{aligned} & 2|B[P \circ \sigma](z) + \Phi_k(R, r, \alpha, \beta)B[P \circ \rho](z)| \\ & \leq \left\{ |\lambda_0| \left| 1 + \Phi_k(R, r, \alpha, \beta) \right| + \frac{1}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| |B[z^n]| \right\} \max_{|z|=k} |P(z)| \\ & \quad + \left\{ |\lambda_0| \left| 1 + \Phi_k(R, r, \alpha, \beta) \right| - \frac{1}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| |B[z^n]| \right\} \min_{|z|=k} |P(z)|. \end{aligned}$$

which is equivalent to inequality (31) and thus completes the proof of theorem 2.7. \square

If we take $\alpha = 0$, we get the following.

Corollary 2.8. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \leq 1$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $R > r \geq k$ and $|z| = 1$,*

$$\begin{aligned} & \left| B[P \circ \sigma](z) + \beta \left(\frac{R+k}{k+r} \right)^n B[P \circ \rho](z) \right| \\ & \leq \frac{1}{2} \left[\left\{ \frac{|B[z^n]|}{k^n} \left| R^n + r^n \beta \left(\frac{R+k}{k+1} \right)^n \right| + \left| 1 + \beta \left(\frac{R+k}{k+1} \right)^n \right| |\lambda_0| \right\} \max_{|z|=k} |B[P](z)| \right. \\ (37) \quad & \left. - \left\{ \frac{|B[z^n]|}{k^n} \left| R^n + r^n \beta \left(\frac{R+k}{k+1} \right)^n \right| - \left| 1 + \beta \left(\frac{R+k}{k+1} \right)^n \right| |\lambda_0| \right\} \min_{|z|=k} |B[P](z)| \right] \end{aligned}$$

where $\sigma(z) = Rz$ and $\rho(z) = rz$.

For $\beta = 0$, Theorem 2.6 reduces to the following result.

Corollary 2.9. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \leq 1$, then for all $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $R > r \geq k$ and $|z| = 1$,*

$$\begin{aligned} & |B[P \circ \sigma](z) - \alpha B[P \circ \rho](z)| \leq \frac{1}{2} \left[\left\{ \frac{|B[z^n]|}{k^n} |R^n - \alpha r^n| + |1 - \alpha| |\lambda_0| \right\} \max_{|z|=k} |P(z)| \right. \\ (38) \quad & \left. - \left\{ \frac{|B[z^n]|}{k^n} |R^n - \alpha r^n| - |1 - \alpha| |\lambda_0| \right\} \min_{|z|=k} |P(z)| \right] \end{aligned}$$

where $\sigma(z) = Rz$ and $\rho(z) = rz$. The result is sharp and extremal polynomial is $P(z) = az^n + b$, $|a| = |b| \neq 0$.

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